Saturated de Rham-Witt with coefficients

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Preamble: This talk is based on my thesis, which is still in progress. It's a hard topic to give a satisfying talk on because the objects I want to talk about have a lot of moving parts, which can be hard to keep in your head. Please stop me and ask if you get confused about anything.

1 Review of the de Rham-Witt complex

Setup: Let k be a perfect field of characteristic p, $W = W(k)$, and $\sigma : W \to W$ the Witt vector Frobenius. Let X be a k-scheme; we will sometimes take $X = \operatorname{Spec} R$.

As Joe explained last month, algebraic de Rham cohomology can be described not only as the hypercohomology of the de Rham complex, but also (in characteristic 0) via the formalism of the infinitesimal site. Crystalline cohomology is ordinarily defined via the crystalline site, which is a souped-up version of the infinitesimal site. The de Rham-Witt complex completes the square below.

cohomology theory	de Rham (char 0)	crystalline
explicit construction	de Rham complex	de Rham-Witt complex
abstract description	infinitesimal site	crystalline site

My project is on the de Rham-Witt complex. It does require a number of inputs from the classical crystalline theory, but since I don't expect everyone to have read Arthur's book (yet), I will do my best to black-box these inputs.

The de Rham-Witt complex of X/k is defined as the initial object in a rather complicated category. Rather than giving its full definition, I will just outline what kinds of structure it

*Notes for a talk in Berkeley's student arithmetic geometry seminar, based on my upcoming thesis.

has, and some of the key conditions we impose. It contains the data:

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\downarrow R & & \downarrow R & & \\
W_2\mathcal{O}_X & \xrightarrow{d} & W_2\Omega^1_X & \xrightarrow{d} & \cdots \\
\downarrow R & & \downarrow R & & \\
W_1\mathcal{O}_X & \xrightarrow{d} & W_1\Omega^1_X & \xrightarrow{d} & \cdots
\end{array}$$

Here each $W_r\Omega^i_X$ is a sheaf of $W_r\mathcal{O}_X$-modules, with $W_r(k)$-linear differentials and vertical quotient maps. (The bottom row is just the de Rham complex of X, and the leftmost column is the sheaf of Witt vectors of $\mathcal{O}_X$.) Additionally, each row has a multiplication map making it a cdga. Finally, each column has maps F going down and V going up, satisfying the following relations:

(a) $FV = VF = p$

(b) $dF = pFd$, $Vd = pdV$, $FdV = d$,

(c) $F(a) = \sigma(a)$ and $V(a) = p\sigma^{-1}(a)$ for $a \in W$,

and various others.

The complex $W\Omega^\bullet_X$ is defined as $\lim_r W_r\Omega^\bullet_X$. The F, V, and d operators and the multiplication map pass to the inverse limit, and they have the same relations as above. Given $W\Omega^\bullet_X$ with all of these operators, we can recover $W_r\Omega^\bullet_X$ as its quotient by the images of V^r and dV^r. In practice, we pass between $W\Omega^\bullet_X$ and $(W_r\Omega^\bullet_X)_r$ more or less freely, but one must be somewhat cautious about what operations do and don't commute with the limit.

Remark 1.1. The condition $dF = pFd$ says that F is a *divided Frobenius*—namely, that the endomorphism given by $\phi = p^nF$ in degree n commutes with d. This is the map that gives rise to the semilinear Frobenius operator ϕ on crystalline cohomology.

Theorem 1.2. *If X/k is smooth, the hypercohomology of the de Rham-Witt complex computes crystalline cohomology. More precisely, we have isomorphisms*

$$Ru_{r*}(\mathcal{O}_{X/W_r,\mathrm{cris}}) \cong W_r\Omega^\bullet_X$$
$$Ru_*(\mathcal{O}_{X/W,\mathrm{cris}}) \cong W\Omega^\bullet_X$$

in $D(\mathrm{Ab}(X))$, where $u_r : (X/W_r)_{\mathrm{cris}} \to X_{\mathrm{Zar}}$ and $u : (X/W)_{\mathrm{cris}} \to X_{\mathrm{Zar}}$ are the usual morphisms of sites. After applying $R^i\Gamma$, these become isomorphisms

$$H^*_{\mathrm{cris}}(X/W_r) \cong \mathbb{H}^*(W_r\Omega^\bullet_X)$$
$$H^*_{\mathrm{cris}}(X/W) \cong \mathbb{H}^*(W\Omega^\bullet_X).$$

2 Recap of BLM

Bhatt-Lurie-Mathew introduces the so-called *saturated de Rham-Witt complex* $\mathcal{W}\Omega^*_X$ of an $\mathbb{F}_p$-scheme X, which agrees with the classical one for X/k smooth, but not in general. (The curly $\mathcal{W}$ distinguishes it from the classical one.) This is constructed affine-locally and over the absolute base $\mathbb{F}_p$, so we will work with an $\mathbb{F}_p$-algebra R instead of a k-scheme X.[1] The key categories involved are as follows.

Definition 2.1. A Dieudonné complex is a complex M^* of abelian groups equipped with an endomorphism $F : M^i \to M^i$ for each i such that $dF = pFd$. The category of Dieudonné complexes is denoted **DC**.

The category **DC** has an important subcategory $\mathbf{DC}_{\mathrm{str}}$, called the category of *strict* Dieudonné complexes, which are particularly nice:

- p-torsionfree
- have a Verschiebung map V satisfying the properties mentioned earlier, and
- complete: $M^* = \lim_r \mathcal{W}_r M^*$, where $\mathcal{W}_r M^* = M^*/(\operatorname{im} V^n + \operatorname{im} dV^n)$.

The inclusion $\mathbf{DC}_{\mathrm{str}} \hookrightarrow \mathbf{DC}$ has a left-adjoint $\mathcal{W}\,\mathrm{Sat}$, called "strictification".

Definition 2.2. 1. The *saturated de Rham-Witt complex* functor $\mathcal{W}\Omega^*_{\_} : \mathbb{F}_p - \mathbf{alg} \to \mathbf{DA}_{\mathrm{str}}$ is defined as the left-adjoint to the functor

$$A^* \mapsto A^0/VA^0,$$

where $\mathbf{DA}_{\mathrm{str}}$ is the category of "strict Dieudonné algebras"; i.e. commutative algebra objects in $\mathbf{DC}_{\mathrm{str}}$ satisfying a few extra (mild) conditions.

2. If X is an $\mathbb{F}_p$-scheme, we let $\mathcal{W}\Omega^*_X$ be the sheaf defined on affines by $R \mapsto \mathcal{W}\Omega^*_R$.

Bhatt-Lurie-Mathew gives two ways to construct $\mathcal{W}\Omega^*_R$, both of which involve writing down certain de Rham complexes, giving them the structure of a Dieudonné complex, and then applying $\mathcal{W}\,\mathrm{Sat}$.

Theorem 2.3. *(BLM, Ogus) Suppose X is an $\mathbb{F}_p$-scheme, smooth over some perfect field k.*

1. *The saturated de Rham-Witt complex $\mathcal{W}\Omega^*_X$ agrees with the classical one $W\Omega^*_{X/k}$.*
2. *Its hypercohomology computes crystalline cohomology.*

(Part 1 is proved in BLM; part 2 follows from part 1 and Illusie's classical comparison, and is also proved independently in the 2020 update to BLM and by Ogus.) However, when X is not smooth over a perfect field, the classical and saturated de Rham-Witt complexes are generally different, and one hopes that that the saturated one should be "better" in some sense.

[1] It turns out that the de Rham-Witt complex doesn't depend on your choice of a base field, provided it's perfect of characteristic p and your scheme is defined over it. This is analogous to the statement that if R is a k-algebra for k perfect, then $\Omega^*_{R/k} = \Omega^*_{R/\mathbb{Z}}$, because if $x \in k$ then $dx = d((x^{1/p})^p) = px^{(p-1)/p}dx^{1/p} = 0$.

3 My project

After the classical de Rham-Witt complex was introduced in the late 1970s, there was a flurry of work to generalize its construction, understand its structure, and study its implications for crystalline cohomology. The goal of my project is to take a baby step towards doing the same within the framework of Bhatt-Lurie-Mathew.

Setup: let R be an $\mathbb{F}_p$-algebra, and $(\mathcal{E}, \phi_{\mathcal{E}})$ a unit-root F-crystal on $\operatorname{Spec} R$. We would like to define a saturated de Rham-Witt complex on $\operatorname{Spec} R$ with coefficients in $\mathcal{E}$. (The classical version of this was done by Étesse in 1987.)

For people who are less familiar, I won't give the full definition of a unit-root F-crystal, but let me say a few words about what they are like. A crystal is a special kind of sheaf on the crystalline site. In particular, it's a sheaf where you can take sections on not just open subschemes of $\operatorname{Spec} R$, but also on nilpotent thickenings of these equipped with divided power structures. For example, we can evaluate $\mathcal{E}$ on $\operatorname{Spec} W_r(R)$, or on $\operatorname{Spec} A/p^r A$ if A is a lift of R with Frobenius. The crystalline structure sheaf $\mathcal{O}_{X/\mathbb{Z}_p}$ is the sheaf $T \mapsto \mathcal{O}_T(T)$.

An F-crystal is a finite locally free $\mathcal{O}_{X/\mathbb{Z}_p}$-module endowed with a semilinear Frobenius endomorphism; it is unit-root if this Frobenius is an isomorphism. In particular, the motivation for studying unit-root F-crystals in this project is that they're more or less the smallest interesting class of F-crystals beyond the trivial crystal. They don't contain much arithmetic content because their slopes are all zero, and they don't contain much geometric content because they're locally free. What they do have is monodromy:

Theorem 3.1. *(Katz, Crew) If X/k is smooth, then there is an equivalence of categories*

$$\{\textit{étale } \mathbb{Z}_p-\textit{local systems on } X\} \leftrightarrow \{\textit{unit-root F-crystals on } X\}$$

given by

$$\mathcal{L} = (\mathcal{L}_n)_n \mapsto (\mathcal{L}_\bullet \otimes_{\mathbb{Z}/p^\bullet\mathbb{Z}} \mathcal{O}_{X/\mathbb{Z}_p}, F = \operatorname{id} \otimes F).$$

3.1 de Rham-Witt modules

What kind of object should $\mathcal{W}\Omega^*_{R,\mathcal{E}}$ be?

- First of all, it should be a strict Dieudonné complex.
- Since $\mathcal{E}$ is a module over the trivial crystal $\mathcal{O}_{\operatorname{Spec} R/\mathbb{Z}_p}$, $\mathcal{W}\Omega^*_{R,\mathcal{E}}$ should be a module over $\mathcal{W}\Omega^*_R = \mathcal{W}\Omega^*_{R,\mathcal{O}}$ in $\mathbf{DC}_{\text{str}}$.
- It should also receive a map from $\mathcal{E}$ in some sense–this reflects the fact that $W\Omega^0_R = W(R) = \lim_r \mathcal{O}_{R/\mathbb{Z}_p}(W_r(R))$.

Definition 3.2. A *de Rham-Witt module over* $(R, \mathcal{E})$ is a collection of the following data: a $\mathcal{W}\Omega^*_R$-module M^* in $\mathbf{DC}_{\text{str}}$, equipped with $W_r(R)$-linear maps $\iota_r : \mathcal{E}(W_r(R)) \to \mathcal{W}_r M^0$ for each r, such that:

1. The ι_r are compatible with the quotient and Frobenius maps on its source and target.
2. (Compatibility with differential) The maps ι_r extend to maps of dg-$\Omega^*_{W_r(R),\gamma}$-modules

$$\iota_r^* : (\mathcal{E}(W_r(R)) \otimes_{W_r(R)} \Omega^*_{W_r(R),\gamma}, \nabla) \to (\mathcal{W}_r\, M^*, d),$$

   where the left side is the *PD-de Rham complex associated to* $\mathcal{E}$. (The target is a graded $\Omega^*_{W_r(R),\gamma}$-module, so there exists a unique such map of graded modules, and we are demanding that it be compatible with the differentials.)

A morphism of de Rham-Witt modules over $(R, \mathcal{E})$ is a morphism $f : M^* \to N^*$ of strict $\mathcal{W}\,\Omega^*_R$-modules such that $\iota_{r,N} = \mathcal{W}_r(f^0) \circ \iota_{r,M}$ for each r. We call the resulting category $\mathrm{dRWM}_{R,\mathcal{E}}$.

Definition 3.3. The *saturated de Rham-Witt complex associated to* $\mathcal{E}$ *over* R, $\mathcal{W}\,\Omega^*_{R,\mathcal{E}}$, is the initial object of $\mathrm{dRWM}_{R,\mathcal{E}}$, assuming this exists.

Goal 3.4. Prove that this always exists, and when R is smooth over a perfect field k it agrees with the classical version $W\Omega^*_{R,\mathcal{E}}$ and computes the cohomology of $\mathcal{E}$.

Useful facts:

1. (Sanity check: the trivial crystal) The saturated de Rham-Witt complex of $\mathcal{E} = \mathcal{O}$ over R is $\mathcal{W}\,\Omega^*_R$, viewed as a module over itself, with suitable maps ι_r. (So $\mathcal{W}\,\Omega^*_R$ satisfies not only BLM's universal property as an algebra, but also our new universal property as a module over itself.)
2. (Functorialities) The category $\mathrm{dRWM}_{R,\mathcal{E}}$ is functorial in R and in $\mathcal{E}$. That is:
   (a) Given a morphism $f : \operatorname{Spec} R' \to \operatorname{Spec} R$ of affine $\mathbb{F}_p$-schemes and $\mathcal{E}$ on $\operatorname{Spec} R$, we have a functor $f_* : \mathrm{dRWM}_{R', f^*_{\mathrm{cris}}\mathcal{E}} \to \mathrm{dRWM}_{R,\mathcal{E}}$.
   (b) Given a morphism $g : \mathcal{E} \to \mathcal{E}'$ of unit-root F-crystals on $\operatorname{Spec} R$, we have a functor $g^* : \mathrm{dRWM}_{R,\mathcal{E}'} \to \mathrm{dRWM}_{R,\mathcal{E}}$.

   Both functorialities act as the identity on the underlying strict Dieudonné complexes, and are defined by pushing forward the module structures and/or composing the ι maps with other maps as necessary.
3. (Insensitivity to nilpotent thickenings) If $\mathcal{E}$ is a unit-root F-crystal on $\operatorname{Spec} R$ and $f : \operatorname{Spec} R_{\mathrm{red}} \hookrightarrow \operatorname{Spec} R$ is the natural closed embedding, then the functor

$$f_* : \mathrm{dRWM}_{R_{\mathrm{red}}, f^*_{\mathrm{cris}}\mathcal{E}} \to \mathrm{dRWM}_{R,\mathcal{E}}$$

   is an equivalence of categories.
4. (Étale base change) If $\mathcal{W}\,\Omega^*_{R,\mathcal{E}}$ exists, then $\mathcal{W}\,\Omega^*_{S,f^*_{\mathrm{cris}}\mathcal{E}}$ exists for every étale map $f : \operatorname{Spec} S \to \operatorname{Spec} R$. Moreover,

$$S \mapsto \mathcal{W}_r\,\Omega^*_{S,f^*_{\mathrm{cris}}\mathcal{E}}$$

   defines a quasicoherent sheaf on the étale site of $\operatorname{Spec} W_r(R)$ (= that of $\operatorname{Spec} R$). This lets us make sense of $\mathcal{W}\,\Omega^*_{X,\mathcal{E}}$ as a sheaf when X is not necessarily affine.

and so on.

Problem: when $\mathcal{E} \neq \mathcal{O}$, there's no obvious construction of an initial de Rham-Witt module. The natural thing to try would be as follows: give $\lim_r(\mathcal{E}(W_r(R)) \otimes_{W_r(R)} \Omega^*_{W_r(R),\gamma})$ the structure of a Dieudonné complex and take its strictification. But I spent an entire summer working on this, and I couldn't make the very first step work: the degree-0 part $\lim_r \mathcal{E}(W_r(R))$ comes with a Frobenius, but I couldn't endow the higher-degree terms with the right divided Frobenius operators.

Instead, as is often the case in crystalline cohomology, we'll have to work with a lift.

3.2 de Rham-Witt lift modules

Setup: Let R be an $\mathbb{F}_p$-algebra and $\mathcal{E}$ a unit-root F-crystal on $\operatorname{Spec} R$, as usual. Suppose A is a p-torsionfree ring with $A/pA \xrightarrow{\sim} R$, and suppose we have a lift of Frobenius $\phi : A \to A$. Set $A_r = A/p^r A$.

Definition 3.5. We define the category $\mathrm{dRWLM}_{A,\mathcal{E}}$ of *de Rham-Witt lift modules* over $(A, \mathcal{E})$ exactly as we did with $\mathrm{dRWM}_{R,\mathcal{E}}$, but replacing $W_r(R)$ with A_r.

This category is less canonical than $\mathrm{dRWM}_{R,\mathcal{E}}$, since it depends on the data of a lift with Frobenius. However, the following two propositions make it useful for constructing our saturated de Rham-Witt complexes.

Proposition 3.6. *The category* $\mathrm{dRWLM}_{A,\mathcal{E}}$ *has an initial object, whose underlying Dieudonné complex is* $\mathcal{W}\operatorname{Sat}(\lim_r(\mathcal{E}(A_r) \otimes_{A_r} \Omega^*_{A_r,\gamma}))$.

Proposition 3.7. *There is a canonical equivalence of categories* $\mathrm{dRWM}_{R,\mathcal{E}} \to \mathrm{dRWLM}_{A,\mathcal{E}}$.

3.3 Main results

Theorem* 3.8. *Suppose R is a k-algebra, and $\mathcal{E}$ is a unit-root F-crystal on* $\operatorname{Spec} R$ *which is defined over a finitely generated subalgebra. Then $\mathcal{W}\Omega^*_{R,\mathcal{E}}$ exists.*

Proof sketch: Reduce to the case where R is reduced and finitely generated over k. In this case, embed $\operatorname{Spec} R$ in $\mathbb{A}^n_k$, and this in $\mathbb{A}^n_W$. Build a p-torsionfree PD-envelope $\operatorname{Spec} \widetilde{D}$ of $\operatorname{Spec} R \hookrightarrow \mathbb{A}^n_W$. Then we have a nilpotent thickening $g : \operatorname{Spec} R \hookrightarrow \operatorname{Spec} \widetilde{D}/p$. Build an F-crystal $\mathcal{F}$ on $\operatorname{Spec} \widetilde{D}/p$ with $g^*_{\text{cris}}\mathcal{F} = \mathcal{E}$. Then we have equivalences of categories

$$\mathrm{dRWM}_{R,\mathcal{E}} \simeq \mathrm{dRWM}_{\widetilde{D}/p,\mathcal{F}} \simeq \mathrm{dRWLM}_{\widetilde{D},\mathcal{F}},$$

and the last of these categories has an initial object.

Future Theorem 3.9. *Suppose X is a smooth k-scheme and $\mathcal{E}$ is a unit-root F-crystal on X. Then:*

1. *$\mathcal{W}\Omega^*_{X,\mathcal{E}}$ agrees with the classical de Rham-Witt complex $W\Omega^*_{X,\mathcal{E}}$, and*

2. $\mathcal{W}\Omega^*_{X,\mathcal{E}}$ *computes the cohomology of* $\mathcal{E}$*.*

Proof sketch: for (1), the case $\mathcal{E} = \mathcal{O}$ is done by Bhatt-Lurie-Mathew. In general, Katz's theorem tells us that $\mathcal{E} = \mathcal{O} \otimes \mathcal{L}$ for some $\mathbb{Z}_p$-local system $\mathcal{L}$, and then we have

$$W\Omega^*_{X,\mathcal{E}} = W\Omega^*_X \otimes \mathcal{L} = \mathcal{W}\Omega^*_X \otimes \mathcal{L} = \mathcal{W}\Omega^*_{X,\mathcal{E}}.$$

The problem is that this really must be interpreted in terms of *strict Dieudonné complexes valued in sheaves*. Up until now, we've done everything affine-locally, and we were always able to simply work with Dieudonné complexes and show after the fact that they define sheaves. Joe Stahl and I have thought a lot about Dieudonné complexes valued in sheaves, but there are still some technical issues to be worked out surrounding the symmetric monoidal structure on strict Dieudonné complexes of sheaves.

As before, (2) follows from (1) and the classical comparison theorem. Another approach I'm working on is to imitate Ogus's proof that $\mathcal{W}\Omega^*_X$ computes the crystalline cohomology of X. Namely, the construction above exhibits $\mathcal{W}\Omega^*_{R,\mathcal{E}}$ as a strictified PD-de Rham complex of $\mathcal{F}$ over the formal PD-thickening $\operatorname{Spec} R \hookrightarrow \operatorname{Spf} \widetilde{D}$; we can compare this to (the strictification of) the classical de Rham complex

$$\mathcal{E}(\widetilde{D})\widehat{\otimes}_A \Omega^*_{A/W}$$

associated to the smooth embedding $\operatorname{Spec} R \hookrightarrow \operatorname{Spec} A$. The latter is known to compute crystalline cohomology by the classical theory.